Fourier expansion along geodesics

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Abstract: A growth estimate on the Fourier coefficients along geodesics for eigenfunctions of the Laplacian is given on compact hyperbolic manifolds. Along the way, a new summation formula is proved.

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Introduction

Let Y be a compact hyperbolic manifold and let c be a closed geodesic in Y. We are interested in the Fourier coefficients

$$c_k(f) = \int_0^1 f(c(l_c t))e^{-2\pi ikt}dt, \qquad k \in \mathbb{Z},$$

of a smooth function $f \in C^\infty(Y)$. Here l_c is the length of c. Under the assumption that f be an eigenfunction of the Laplace operator on Y with eigenvalue μ, one can relate c_k to an intertwining integral $I_k^\mu(f)$, which depends on μ and f, but not on c. Namely, there is an *automorphic coefficient* $a_k \in \mathbb{C}$ such that

$$c_k = a_k \, I_k^\mu(f).$$

The sequences (c_k) and $(I_k^\mu(f))$ are rapidly decreasing, so there is no a priori knowledge on the growth of (a_k). Surprisingly, it is possible to prove the following bounds, which constitute the main result of the paper.

If $\dim Y = 2$, then, as $T \to \infty$,

$$\sum_{\substack{k\in\mathbb{Z} \\ |k|\le T}} |a_k|^2 = O(T^{1/2}).$$

If $\dim Y > 2$, then

$$\sum_{k\in\mathbb{Z}} |a_k|^2 \ < \ \infty.$$

The proof uses two expicit summation formulae (Theorems 4.1 and 4.3), which are presented as results of own interest. The proof of these two relies on the uniqueness of intertwining functionals (section 2) and the uniqueness of trilinear products [2]. The idea that uniqueness of trilinear products together with explicit formulae can be used to derive growth estimates is due to Bernstein and Reznikov [1], the idea that this can be used for Fourier expansions along geodesics was formulated in [10], the arguments in section 3 are inspired by similar ideas in that paper.

We will explain the construction of the factors a_k in a bit more detail. Let X be the universal covering of Y and Γ its fundamental group. Then Γ acts on X by isometries and Y is the quotient $\Gamma\backslash X$. So Γ injects into the isometry group G of X, which acts transitively on X, so $X \cong G/K$ for a

subgroup K, which turns out to be compact. Let (π, V_π) be an irreducible unitary representation of G and let $\eta : V_\pi \to L^2(\Gamma\backslash G)$ be an isometric linear G-map. Let $P_K : L^2(\Gamma\backslash G) \to L^2(\Gamma\backslash G)^K = L^2(\Gamma\backslash G/K) = L^2(Y)$ denote the orthogonal projection onto the subspace of K-invariants. Demanding that $f \in C^\infty(Y)$ be an eigenfunction of the Laplacian amounts to the same as demanding f to lie in the image of $P_K \circ \eta$ for some π and η. The functional $I_k^\gamma = c_k \circ P_K \circ \eta$ on V_π then has an intertwining property with respect to a split torus A inside G. By a uniqueness result, proven in section 2, this implies that I_k^γ is a multiple of a canonical intertwiner I_k^π on V_π, which we named I_k^μ above. So we get the existence of a factor $a_k \in \mathbb{C}$ with $I_k^\gamma = a_k I_k^\pi = a_k I_k^\mu$ as above.

1 Generalised period integrals

Let d be an integer ≥ 2 and let Y be an orientable compact hyperbolic manifold of dimension d. For a closed geodesic c in Y let $l(c)$ denote its length. The period integral $I_c(f) = \int_0^{l(c)} f(c(t))\, dt$ is the zeroth coefficient of the Fourier-expansion of the function $t \mapsto f(c(t))$. Therefore the higher coefficients can be viewed as "generalised period integrals". We compute this expansion by temporarily turning to the more general setting of functions on the sphere-bundle SY over Y. In the SY, the geodesic c lifts to a closed orbit, again denoted c, of the geodesic flow ϕ_t. For a given point $x_0 \in c$ one is interested in the Fourier-expansion of the function $t \mapsto f(\phi_t x_0)$, where now f is in $C^\infty(SY)$.

Let X be the universal covering of Y and let G be the group of orientation preserving isometries of X if $d \geq 3$, and for $d = 2$ let G be the group of all isometries of X. Then G is isomorphic to the connected component $\mathrm{SO}(d,1)^0$ of the special orthogonal group $\mathrm{SO}(d,1)$ if $d \geq 3$, and for $d = 2$ one has $g \cong \mathrm{PGL}_2(\mathbb{R})$. The group G acts transitively on X, which thereby can be identified with G/K, where $K \cong \mathrm{SO}(d)$ if $d \geq 3$, and $K \cong \mathrm{O}(2)$ if $d = 2$, is the maximal compact subgroup of G. The Riemannian metric on X determines an invariant symmetric bilinear form b on the real Lie algebra $\mathfrak{g}_\mathbb{R}$ of G. Let $\mathfrak{k}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ be the Lie algebra of K. Then b is negative definite on $\mathfrak{k}_\mathbb{R}$ and positive definite on its orthogonal complement $\mathfrak{p}_\mathbb{R}$. Let $\mathfrak{a}_\mathbb{R}$ be a one-dimensional subspace of $\mathfrak{p}_\mathbb{R}$, and let $A = \exp(\mathfrak{a}_\mathbb{R})$ be the corresponding

subgroup of G. Then A is closed and non-compact and its centraliser equals AM, where M is the centraliser of A in K. Let $\mathfrak{a}$ be the complexification of $\mathfrak{a}_\mathbb{R}$ and $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Then $\mathfrak{a}^*$ can be identified with the set of continuous homomorphisms from A to $\mathbb{C}^\times$. For $\lambda \in \mathfrak{a}^*$ we write $a \mapsto a^\lambda = e^{\lambda(\log a)}$ for the corresponding homomorphism. It is known that $G/M \to G/K$ can be identified with the sphere-bundle of $X = G/K$ in a way that the geodesic flow is given by

$$\phi_t(gM) = g \exp(tH_1)M,$$

where H_1 is a fixed element of $\mathfrak{a}_\mathbb{R}$ with $b(H_1, H_1) = 1$.

The fundamental group Γ of Y acts on X by deck-transformations and can thus be viewed as a uniform lattice in G. It follows that $Y = \Gamma\backslash X = \Gamma\backslash G/K$. The sphere bundle SY equals $\Gamma\backslash G/M$.

A closed geodesic c in Y gives rise to a conjugacy class $[\gamma]$ in Γ of elements which "close" c. Any such $\gamma \in \Gamma$ is *hyperbolic* in the sense that it is conjugate in G to an element of the form $a_\gamma m_\gamma \in AM$ with $a_\gamma \neq 1$. The characters of the compact abelian group $A/\langle a_\gamma \rangle$ are given by those $\lambda \in \mathfrak{a}^*$ with $a_\gamma^\lambda = 1$, i.e., $\lambda(\log a_\gamma) \in 2\pi i\mathbb{Z}$. Let λ_γ be the unique element of $\mathfrak{a}_\mathbb{R}^*$ with $\lambda_\gamma(\log a_\gamma) = 2\pi i$. Then $\widehat{A/\langle a_\gamma \rangle} = \mathbb{Z}\lambda_\gamma$.

Fix $k \in \mathbb{Z}$ Fix an element $x_\gamma \in G$ with $\gamma = x_\gamma a_\gamma m_\gamma x_\gamma^{-1}$ and let

$$\begin{array}{rcl} I_k^\gamma : C^\infty(\Gamma\backslash G/M) & \to & \mathbb{C} \\ \varphi & \mapsto & \dfrac{1}{l(\gamma)} \displaystyle\int_{A/\langle a_\gamma \rangle} \varphi(x_\gamma a) a^{-k\lambda_\gamma}\, da, \end{array}$$

where the Haar measure da is determined by the metric and $l(\gamma)$ equals $|\log a_\gamma| = \mathrm{vol}(A/\langle a_\gamma \rangle$. Note that I_k^γ depends on the choice of x_γ. Geometrically, this corresponds to choosing a base-point on the closed orbit c. This dependence is not severe, as x_γ is determined up to multiplication from the right by elements of AM_{m_γ}, where M_{m_γ} is the centraliser in M of m_γ. If we replace x_γ by $x_\gamma a_0 m_0$, then I_k^γ is replaced by $a_0^{k\lambda_\gamma} I_k^\gamma$. So in particular, the map $|I_k^\gamma|$ is uniquely determined by k and γ. Further, if γ is replaced by a Γ-conjugate, say $\gamma' = \sigma\gamma\sigma^{-1}$ then one can choose $x_{\gamma'}$ to be equal to σx_γ and with this choice one gets $I_k^{\gamma'} = I_k^\gamma$.

The space $C^\infty(\Gamma\backslash G/M)$ can be viewed as the space $C^\infty(\Gamma\backslash G)^M$ of M-invariants in $C^\infty(\Gamma\backslash G)$. The volume element on X determines a Haar mea-

sure dg on G. Let R denote the unitary G-representation on $L^2(\Gamma\backslash G)$ given by right translations. As Γ is co-compact,

$$L^2(\Gamma\backslash G) \;\cong\; \bigoplus_{\pi\in\hat{G}} N_\Gamma(\pi)\pi,$$

where the sum runs over the unitary dual $\hat{G}$ of G and the multiplicities $N_\Gamma(\pi)$ are finite. Here and later we understand the direct sum to be a completed direct sum in the appropriate topology. Then

$$C^\infty(\Gamma\backslash G) = L^2(\Gamma\backslash G)^\infty = \bigoplus_{\pi\in\hat{G}} N_\Gamma(\pi)\pi^\infty,$$

where π^∞ is the representation on the Fréchet space of smooth vectors. Further,

$$C^\infty(\Gamma\backslash G/M) = C^\infty(\Gamma\backslash G)^M = \bigoplus_{\pi\in\hat{G}} N_\Gamma(\pi)\left(\pi^\infty\right)^M.$$

Note that, as M centralises A, the representation $R|_A$ can be pushed down to $C^\infty(\Gamma\backslash G/M)$. The linear functional I_k^γ satisfies

$$I_k^\gamma(R(a)\varphi) = a^{k\lambda_\gamma} I_k^\gamma(\varphi)$$

for every $a \in A$. This means that I_k^γ is an intertwining functional.

2 Intertwining functionals

Let $P = MAN$ be a parabolic with split component A. Let $\mathfrak{m}_\mathbb{R}, \mathfrak{n}_\mathbb{R}$ be the real Lie algebras of M and N and let $\mathfrak{m}, \mathfrak{n}$ be their complexifications. The modular shift $\rho \in \mathfrak{a}^*$ is defined by the equation $a^{2\rho} = \det(a|\mathfrak{n})$, where $\mathfrak{n}$ is the complexified Lie algebra of N on which A acts by the adjoint representation. We establish an ordering on the one dimensional real vector space $\mathfrak{a}_\mathbb{R}$ by $t\rho > 0 \Leftrightarrow t > 0$.

Let (σ, V_σ) be a finite dimensional irreducible representation of M and let λ be an element of the dual space $\mathfrak{a}^*$. Let $\pi_{\sigma,\lambda}$ be the corresponding principal series representation, which we normalise to live on functions $f : G \to V_\sigma$ satisfying $f(manx) = a^{\lambda+\rho}\sigma(m)f(x)$. Let $V_{\sigma,\lambda}$ be the space of $\pi_{\sigma,\lambda}$ and let

$V_{\sigma,\lambda}^\infty$ be the space of smooth vectors in it. These can be viewed as smooth sections of the vector bundle $E_{\sigma,\lambda}$ over $P\backslash G$ given by the P-representation $(man) \mapsto a^{\lambda+\rho}\sigma(m)$. Note that restriction of functions to K identifies $V_{\sigma,\lambda}^\infty$ with the space of all smooth sections of the homogeneous vector bundle E_σ on $M\backslash K$ given by the representation σ. These can be interpreted as smooth functions $f : K \to V_\sigma$ with $f(mk) = \sigma(m)f(k)$ for $m \in M$, $k \in K$.

Let $k \in \mathbb{Z}$. A continuous linear functional $l : \left(V_{\sigma,\lambda}^\infty\right)^M \to \mathbb{C}$ is called a *k-intertwiner*, if

$$l(\pi(a)v) = a^{k\lambda_\gamma} l(v)$$

holds for every $a \in A$ and $v \in \left(V_{\sigma,\lambda}^\infty\right)^M$. Let $V_{\sigma,\lambda}^\infty(k)$ be the space of all k-intertwiners.

Let w_0 be a representative in K of the non-trivial element of the Weyl group $W(G,A)$ and let $n_0 \in N$ be a fixed element different from 1.

Proposition 2.1 *Assume throughout that* $\mathrm{Re}(\lambda) > -\rho$. *Then we have*

$$\dim V_{\sigma,\lambda}^\infty(k) \;=\; \dim V_\sigma.$$

More precisely, let $l \in V_\sigma^*$. Then the integral

$$I_{k,l}^{\sigma,\lambda}(f) \;=\; \int_A l(f(w_0 n_0 a)) a^{-k\lambda_\gamma}\, da$$

converges for every $f \in \left(V_{\sigma,\lambda}^\infty\right)^M$. The map $l \mapsto I_{k,l}^{\sigma,\lambda}$ is a linear bijection

$$V_\sigma^* \xrightarrow{\cong} V_{\sigma,\lambda}^\infty(k).$$

Dependence on the choice of n_0: If we replace n_0 by another non-trivial element n_0', then there exists $a_0 m_0 \in AM$ such that $n_0' = a_0 m_0 n_0 (a_0 m_0)^{-1}$. Then $I_{k,l}^{\sigma,\lambda}$ gets replaced with $a_0^{-\lambda-\rho-k\lambda_\gamma} I_{k,l'}^{\sigma,\lambda}$, where $l' = l \circ \sigma(w_0 m_0 w_0^{-1})$.

Proof: The base space $P\backslash G$ of the bundle $E_{\sigma,\lambda}$ consists of three orbits under the group AM, namely the open orbit $[w_0 n_0]$, and the two closed orbits $[1]$, $[w_0]$ which are indeed points. We consider the closed orbits first. Fix $l \in V_\sigma^*$ and let T_0 denote the distribution

$$T_0(f) = l(f(1)).$$

Then $T_0 \circ R(a) = a^{\lambda+\rho} T_0$, so T_0 is a k-intertwiner for $a^{k\lambda_\gamma} = a^{\lambda+\rho}$. Let $\bar{N} = \theta(N)$ and let $\bar{\mathfrak{n}}_{\mathbb{R}}$ be its Lie algebra. Then the tangent space of $P\backslash G$ at the unit is isomorphic to $\bar{n}_{\mathbb{R}}$. So for $X \in \bar{\mathfrak{n}}_{\mathbb{R}}$ we set

$$T_X(f) = l(Xf(1)).$$

Then $T_X \circ R(a) = a^{\lambda+\rho+\alpha_0}$, where α_0 is the positive root, i.e., $\alpha_0 = \frac{2}{d-1}\rho$. For $X_1, \dots X_k \in \bar{n}_{\mathbb{R}}$ set

$$T_{X_1 \dots X_k}(f) = l(X_1 \dots X_k f(1)).$$

Then $T_{X_1 \dots X_k} \circ R(a) = a^{\lambda+\rho+\frac{2k}{d-1}\rho}$. Since these span the space of all distributions supported at 1 we see that we only get a k-intertwiner supported on 1 if $a^{k\lambda_\gamma} = a^{\lambda+\rho+\frac{2k}{d-1}\rho}$ for some $k \geq 0$. The latter condition implies $\mathrm{Re}(\lambda) \leq -\rho$, which is outside the range of the proposition.

We turn to the other closed orbit $[w_0]$. In this case, let $X_1, \dots, X_k \in \mathfrak{n}_{\mathbb{R}}$ and define

$$S_{X_1 \dots X_k}(f) = l(X_1 \dots X_k f(w_0)).$$

Then $S_{X_1 \dots X_k} \circ R(a) = a^{-\lambda-\rho-\frac{2k}{d-1}\rho}$ and likewise, all intertwiners supported on $[w_0]$ lie at $\mathrm{Re}(\lambda) \leq -\rho$. Since we assume $\mathrm{Re}(\lambda) > -\rho$, this means that every k-intertwiner which vanishes on the open orbit is zero. As an M-space, the open orbit $[w_0 n_0]$ is isomorphic to $M/M_0 \times A$, where $M_0 \cong \mathrm{SO}(d-2) \subset \mathrm{SO}(d-1) \cong M$. So the M-invariant sections of $E_{\sigma,\lambda}|_{[w_0 n_0]}$ may be viewed as the sections of a vector bundle F on $[w_0 n_0]/M \cong A$. By Lemma 2.2 of [2] every equivariant distribution on F is given by a smooth section which must be A-invariant, hence is uniquely determined by its value at one point. So the space of such distributions is in bijection with the fibre V_σ^*. Since it is not clear a priori that any intertwiner defined on the open orbit indeed extends to the whole space $P\backslash G$, this argument only shows $\dim V_{\sigma,\lambda}^\infty(k) \leq \dim V_\sigma$. For the other direction we need to show the convergence of the integral $I_{k,l}^{\sigma,\lambda}$. So let $f \in V_{\sigma,\lambda}^\infty$ and compute formally

$$I_{k,l}^{\sigma,\lambda}(f) = \int_A \underline{a}(w_0 n_0 a)^{\lambda+\rho} l(f(\underline{k}(w_0 n_0 a))) a^{-k\lambda_\gamma}\, da,$$

where $\underline{a}$ and $\underline{k}$ are the projections $G \to K$ and $G \to A$ of the ANK-Iwasawa decomposition of G. Now $l(f(\underline{k}(w_0 n_0 a)))a^{k\lambda_\gamma}$ is bounded, so we only need to show the convergence of

$$\int_A \underline{a}(w_0 n_0 a)^{\mathrm{Re}(\lambda)+\rho} da = \int_A a^{-\mathrm{Re}(\lambda)-\rho} \underline{a}(w_0 n_0^a)^{\mathrm{Re}(\lambda)+\rho}\, da,$$

where $n_0^a = a^{-1}n_0a$. By the explicit expressions derived in [2] this equals

$$\int_{\mathbb{R}} \left(\frac{1}{e^t+e^{-t}}\right)^{\frac{d-1}{2}(1+\mathrm{Re}(\bar{\lambda}))} dt,$$

where $\lambda = \bar{\lambda}\rho$, with $\mathrm{Re}(\bar{\lambda}) > -1$. The convergence is clear.

Finally we have to show the injectivity of $l \mapsto I_{k,l}^{\sigma,\lambda}$. For this it suffices to consider sections f which are M-invariant and vanish in a neighbourhood of $\{1, w_0\}$. Since no two points of the form w_0n_0a are M-conjugate, we can prescribe the value of $f(w_0n_0a)$ arbitrarily as long as it is smooth in a and compactly supported. This implies injectivity. $\square$

A sequence $(c_j)_{j\in\mathbb{N}}$ of complex numbers is said to be *of moderate growth*, if there exist $N \in \mathbb{N}$ such that

$$|c_j| = O(j^N),$$

as $j \to \infty$. The sequence is called *rapidly decreasing*, if for every $N \in \mathbb{N}$ one has

$$|c_j| = O(j^{-N})$$

as $j \to \infty$. The product of two moderately growing sequences is moderately growing and the product of a moderately growing sequence and a rapidly decreasing sequence is rapidly decreasing.

Proposition 2.2 *Let* $\mathrm{Re}(\lambda) > -\rho$ *and* σ *be given. Then for any* $l \in V_\sigma^*$ *and* $f \in (V_{\sigma,\lambda}^\infty)^M$ *the sequence*

$$\left(I_{k,l}^{\sigma,\lambda}\right)_{k\in\mathbb{Z}}$$

is rapidly decreasing.

Proof: By definition it is clear that the sequence $a_k = I_{k,l}^{\sigma,\lambda}(f)$ is bounded, as λ_γ is purely imaginary. Next, using integration by parts one sees that for $N \in \mathbb{N}$,

$$\begin{aligned} k^N a_k &= \int_{\mathbb{R}} l(f(w_0n_0\exp(tH))k^N e^{-kt\lambda_\gamma(H)}dt \\ &= \overline{\lambda_\gamma(H)}^N \int_{\mathbb{R}} l(H^N f(w_0n_0\exp(tH))e^{-kt\lambda_\gamma(H)}dt \\ &= \overline{\lambda_\gamma(H)}^N I_{k,l}^{\sigma,\lambda}(H^N f). \end{aligned}$$

This, again, is bounded and we get the claim. □

If σ is the trivial representation, then $V_\sigma = \mathbb{C}$ and we can choose l to be the identity. In that case we write I_k^λ for $I_{k,l}^{\sigma,\lambda}$.

An irreducible unitary representation (π, V_π) is called a *class one* or *spherical* representation, if the space of K-invariant vectors V_π^K is non-zero. In that case one has $\dim V_\pi^K = 1$ (see [3]). Let $\hat{G}^K$ denote the set of all $\pi \in \hat{G}$ which are spherical.

Proposition 2.3 *Let $\pi \in \hat{G}^K$. If $\pi \neq 1$, then*

$$\dim V_\pi^\infty(k) = 1.$$

If $\pi = 1$, then $V_\pi^\infty(k)$ is zero unless $k = 0$ in which case it is one dimensional. For every $\pi \in \hat{G}^K \setminus \{1\}$ there is a canonical generator I_k^π given by the integral of Proposition 2.1.

Proof: Since $\hat{G}^K \setminus \{1\}$ consists of the principal series $\pi_{1,\lambda}$ for $\lambda \in i\mathfrak{a}_\mathbb{R}$ and the complementary series $\pi_{1,\lambda}$ for $0 < \lambda < \rho$ the claim follows from Proposition 2.1. □

By an *automorphic representation* (π, V_π, η) we mean an irreducible unitary representation (π, V_π) of G together with an isometric G-equivariant linear map $\eta\colon V_\pi \to L^2(\Gamma\backslash G)$. Then η maps the space V_π^∞ of smooth vectors into $C^\infty(\Gamma\backslash G)$. The automorphic representation (π, V_π, η) is called a *spherical automorphic representation*, if π is spherical. In that case, by Proposition 2.3 it follows that for γ as in section 1 and $k \in \mathbb{Z}$ there exists a unique complex number $a_k^{\eta,\gamma}$ such that

$$I_k^\gamma(\eta(v)) = a_k^{\eta,\gamma} I_k^\pi(v)$$

for every $v \in V_\pi^\infty$. The factor $a_k^{\eta,\gamma}$ carries the "automorphic" information.

Since both sequences, $I_k^\gamma(\eta(v))$ and $I_k^\pi(v)$ are rapidly decreasing, there is no a priori knowledge on the growth of $(a_k^{\eta,\lambda})_k$. All the more striking is the following theorem, which constitutes the main result of this paper.

Theorem 2.4 *If $d = 2$, then, as $T \to \infty$,*

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \le T}} |a_k^{\eta,\gamma}|^2 = O(T^{1/2}).$$

If $d \geq 3$, then we have

$$\sum_{k \in \mathbb{Z}} |a_k^{\eta,\gamma}|^2 \ < \ \infty.$$

The proof extends over the next two sections.

3 Triple products

Let (π, V_π, η) be a spherical automorphic representation. Since π is unitary, there is an anti-linear isomorphism to the dual $c : V_\pi \to V_\pi^*$. Let $\breve{\pi}$ denote the dual representation on $V_\pi^* = V_{\breve{\pi}}$. Let $\bar{\cdot}$ be the complex conjugation on $L^2(\Gamma \backslash G)$ and let $\breve{\eta}$ be the composition of the maps

$$V_{\breve{\pi}} \xrightarrow{\quad c^{-1} \quad} V_\pi \xrightarrow{\quad \eta \quad} L^2(\Gamma \backslash G) \xrightarrow{\ \bar{\cdot}\ } L^2(\Gamma \backslash G).$$

Then $\breve{\eta}$ is a G-equivariant linear isometry of $V_{\breve{\pi}}$ into $L^2(\Gamma \backslash G)$, so $(\breve{\pi}, V_{\breve{\pi}}, \breve{\eta})$ is an automorphic representation as well.

Let $\Delta : \Gamma \backslash G \to \Gamma \backslash G \times \Gamma \backslash G$ be the diagonal map. Let $\Delta^* : C^\infty(\Gamma \backslash G \times \Gamma \backslash G) \to C^\infty(\Gamma \backslash G)$ be the corresponding pullback map and let $E = V_\pi^\infty \hat{\otimes} V_{\breve{\pi}}^\infty$, where $\hat{\otimes}$ denotes the projective completion of the algebraic tensor product. Let η_E denote the natural embedding $E \to C^\infty(\Gamma \backslash G) \hat{\otimes} C^\infty(\Gamma \backslash G) \cong C^\infty(\Gamma \backslash G \times \Gamma \backslash G)$. For γ as in the first section we get an induced functional on E,

$$l_{\Delta(\gamma)} = I_0^\gamma \circ \Delta^* \circ \eta_E.$$

In other words, for $w \in E$ we have

$$l_{\Delta(\gamma)}(w) = \frac{1}{l(\gamma)} \int_{A/\langle a_\gamma \rangle} \eta_E(w)(x_\gamma a, x_\gamma a)\, da.$$

This has the Fourier series expansion,

$$\begin{aligned} l_{\Delta(\gamma)}(w) &= \sum_{k\in\mathbb{Z}} I_k^\gamma \otimes I_{-k}^\gamma(w) \\ &= \sum_{k\in\mathbb{Z}} a_k^{\eta,\gamma} a_{-k}^{\check{\eta},\gamma}\, I_k^\pi \otimes I_{-k}^{\check{\pi}}(w) \\ &= \sum_{k\in\mathbb{Z}} |a_k^{\eta,\gamma}|^2\, \hat{w}(k,-k), \end{aligned}$$

where the last line defines $\hat{w}$ and we have used the fact that $a_{-k}^{\check{\eta},\gamma} = \overline{a_k^{\eta,\gamma}}$.

Let (τ, V_τ) be another element of $\hat{G}^K$. According to [2] there is a canonical G-invariant continuous functional

$$T_\tau^{\mathrm{st}} : E \,\hat{\otimes} V_{\check{\tau}}^\infty \;\to\; \mathbb{C},$$

and any other such functional is a scalar multiple of T_τ^{st}. This induces a canonical G-equivariant continuous map

$$T_\tau^{\mathrm{st}} : E \;\to\; V_\tau^\infty.$$

On the other hand we have $\Delta^* \circ \eta_E \colon E \to L^2(\Gamma\backslash G)^\infty$. For an automorphic representation $(\tau, V_\tau, \eta_\tau)$ we have an orthogonal projection $\mathrm{Pr}_{\eta_\tau} : L^2(\Gamma\backslash G) \to V_\tau$ and thus we get a map

$$T_{\eta_\tau}^{\mathrm{aut}} = \mathrm{Pr}_{\eta_\tau} \circ \Delta^* \circ \eta_E$$

from E to V_τ^∞. Hence there is a coefficient $c(\eta, \eta_\tau) \in \mathbb{C}$ such that

$$T_{\eta_\tau}^{\mathrm{aut}} = c(\eta, \eta_\tau) T_\tau^{\mathrm{st}}.$$

Fix a complete family (η_j) of normalised, pairwise orthogonal spherical automorphic representations $\eta_j : \pi_j \to L^2(\Gamma\backslash G)$. Then the spectral expansion of $\Delta^* \circ \eta_E$ is

$$\Delta^* \circ \eta_E = \sum_j c(\eta, \eta_j) T_{\pi_j}^{\mathrm{st}}.$$

And hence, for $w \in E$,

$$l_{\Delta(\gamma)}(w) = \sum_j c(\eta, \eta_j) a_0^{\eta_j,\gamma} I_0^{\pi_j}(T_{\pi_j}^{\mathrm{st}}(w)).$$

So we conclude

Lemma 3.1

$$\sum_{k\in\mathbb{Z}} |a_k^{\eta,\gamma}|^2\, \hat{w}(k,-k) \;=\; \sum_j c(\eta,\eta_j) a_0^{\eta_j,\gamma} I_0^{\pi_j}(T_{\pi_j}^{\mathrm{st}}(w)).$$

4 Proof of Theorem 2.4

We want to make the formula in Lemma 3.1 more explicit. For this let $\alpha \in \mathfrak{a}^*$ be the positive root. Let $H_1 \in \mathfrak{a}$ be the unique element with $\alpha(H_1) = 1$. We scale the metric b in such a way that $b(H_1, H_1) = 1$ holds. Fix an isomorphism $n : \mathbb{R}^{d-1} \to N$ such that the M-action on N by conjugation translates into the action of the group $\mathrm{SO}(d-1)$ on $\mathbb{R}^{d-1}$. For $x \in \mathbb{R}^{d-1}$ let $|x|$ denote the euclidean norm of x.

For $\varphi \in C_c^\infty(\mathbb{R})$ we set

$$f_\varphi(w_0 n(x)) = \varphi(\log|x|).$$

For given $\lambda \in \mathfrak{a}^*$ the function f_φ extends uniquely to an element of V_λ^∞. For $a = \exp(tH_1) \in A$ we have $a^\lambda = e^{b(\lambda,\alpha)t}$. We define $\tilde{\lambda} = \frac{b(\lambda,\alpha)}{2\pi i}$, so that $a^\lambda = e^{2\pi i\tilde{\lambda}t}$. We normalise the Haar measure da on A as usual so that $\int_A g(a)da = \int_{\mathbb{R}} g(\exp(tH_1)dt$. Assuming that $n_0 = n(x_0)$ for some $x_0 \in \mathbb{R}^{d-1}$ with $|x_0| = 1$ we compute for $k \in \mathbb{Z}$,

$$\begin{aligned} I_k^\lambda(f_\varphi) &= \int_A f_\varphi(w_0 n_0^a) a^{-\lambda-\rho-k\lambda_\gamma} da \\ &= \int_{\mathbb{R}} \varphi(t)\, e^{-t(2\pi i\tilde{\lambda}+\frac{d-1}{2}+2\pi i k\tilde{\lambda}_\gamma)} dt. \end{aligned}$$

Likewise, for $\phi \in C_c^\infty(\mathbb{R}^2)$ let $w_\phi \in E$ be defined by $w_\phi(w_0 n(x), w_0 n(y)) = \phi(\log|x|, \log|y|)$. We get

$$\hat{w}_\phi(k,-k) = \int_{\mathbb{R}^2} \phi(t,s)\, e^{(s-t)(2\pi i\tilde{\lambda}+2\pi i k\tilde{\lambda}_\gamma)-(s+t)\frac{d-1}{2}} dt\, ds.$$

With $w = w_\phi$ as above, we want to compute $I_0^{\pi_j}(T_{\pi_j}^{\mathrm{st}}(w))$. Recall that the

functional $T^{\rm st}:\pi\otimes\breve\pi\otimes\breve\pi_j\to\mathbb{C}$ maps a given φ to

$$\begin{aligned} T^{\rm st}(\varphi) &= \int_G \varphi(w_0n_0y.w_0y,y)\,dy \\ &= \int_{ANK} w(w_0n_0an,w_0an)f(ank)\,da\,dn\,dk, \end{aligned}$$

where we have assumed $\varphi = w\otimes f$ for some $f\in\breve\pi_j$. The induced map $T^{\rm st}_{\pi_j}:E\to\pi_j$ is defined via the pairing

$$\breve\pi_j\otimes\pi_j = \pi_{-\lambda_j}\otimes\pi_{\lambda_j} \;\to\; \mathbb{C}$$

given by

$$(f\otimes h)\;\mapsto\;\int_K f(k)h(k)\,dk.$$

The resulting map $T^{\rm st}_{\pi_j}:\pi\otimes\breve\pi\to\pi_j$ maps w into the one-dimensional space of K-invariants in π_j. The restriction of $T^{\rm st}_{\pi_j}(w)$ to K therefore is a constant function taking the value

$$\begin{aligned} R_\phi(\lambda_j) &= \int_{AN} w(w_0n_0an,w_0an)a^{-\lambda_j+\rho}\,da\,dn \\ &= \int_{AN} w(w_0n_0^an,w_0n)a^{-\lambda_j-\rho}\,da\,dn. \end{aligned}$$

We temporarily write $f = T^{\rm st}_{\pi_j}(w)\in\pi^K_{\lambda_j}$. Then $f(k)=R_\phi(\lambda_j)$ and so

$$f(w_0n(x)) = R_\phi(\lambda_j)\underline{a}(w_0n(x))^{\lambda_j+\rho}.$$

So we can compute,

$$\begin{aligned} I_0^{\pi_j}(T^{\rm st}_{\pi_j}(w)) &= \int_A f(w_0n_0^a)a^{-\lambda_j-\rho}da \\ &= \int_A f(w_0n(a^{-\alpha}x_0))a^{-\lambda_j-\rho}da \\ &= \int_{\mathbb{R}} f(w_0n(-tx_0))e^{-t(2\pi i\tilde\lambda_j+\frac{d-1}{2})}dt \\ &= \frac{R_\phi(\lambda_j)}{2}\int_{\mathbb{R}}\frac{e^{-x\frac12(2\pi i\tilde\lambda_j+\frac{d-1}{2})}}{(1+e^{-x})^{2\pi i\tilde\lambda_j+\frac{d-1}{2}}}dt \\ &= R_\phi(\lambda_j)\,\beta(\tilde\lambda_j), \end{aligned}$$

where $\beta(t)$ is the Schwartz function

$$\begin{aligned}\beta(t) &= \int_{\mathbb{R}} (e^x + e^{-x})^{-2\pi i t - \frac{d-1}{2}} dx \\ &= \frac{1}{2} \frac{\Gamma\left(\frac{4\pi i t + d - 1}{4}\right)^2}{\Gamma\left(\frac{4\pi i t + d - 1}{2}\right)}.\end{aligned}$$

(See [4] 3.313.2.)

It remains to compute $R_\phi(\lambda_j)$. We get

$$\begin{aligned}R_\phi(\lambda_j) &= \int_{AN} w_\phi(w_0 n_0^a n, w_0 n) a^{-\lambda_j - \rho}\, da\, dn \\ &= \int_{\mathbb{R}\times\mathbb{R}^{d-1}} \phi(\log|e^{-t}x_0 + x|, \log|x|) e^{-t(2\pi i \tilde{\lambda}_j + \frac{d-1}{2})} dt\, dx.\end{aligned}$$

Choosing x_0 to be $(1, 0, \ldots, 0)^t \in \mathbb{R}^{d-1}$ we see that $R_\phi(\lambda_j)$ equals

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} \phi\left(\frac{1}{2}\log\left((x_1 + e^{-t})^2 + x_2^2 + \cdots + x_{d-1}^2\right), \frac{1}{2}\log\left(x_1^2 + \cdots + x_{d-1}^2\right)\right) \\ \times \quad e^{-t(2\pi i \tilde{\lambda}_j + \frac{d-1}{2})} dt\, dx.$$

Being the Fourier transform of a Schwartz-function, the function R_ϕ is rapidly decreasing.

4.1 The summation formula in the case $d = 2$

The following summation formula is used in our proof, but is also of independent interest.

Theorem 4.1 *(Summation Formula) Let ϕ be a smooth function of compact support on $\mathbb{R}^2$. For $d = 2$ the expression*

$$\sum_{k\in\mathbb{Z}} |a_k^{\eta,\gamma}|^2 \int_{\mathbb{R}^2} \phi(s,t) e^{-2\pi i (s-t)(\tilde{\lambda} + k\tilde{\lambda}_\gamma) - (s+t)\frac{1}{2}} dt\, ds$$

equals

$$2\sum_{j=1}^{\infty} c(\eta, \eta_j) a_0^{\eta_j,\gamma} \beta(\tilde{\lambda}_j) \int_{\mathbb{R}^2} \phi(a,b) e^{a+b} \left(\left(e^a + e^b\right)^{2\pi i \tilde{\lambda}_j - \frac{1}{2}} + |e^a - e^b|^{2\pi i \tilde{\lambda}_j - \frac{1}{2}}\right) da\, db.$$

Proof: In the case $d = 2$ the number $R_\phi(\lambda_j)$ equals

$$\int_{\mathbb{R}^2} \phi(\log|x + e^{-t}|, \log|x|) e^{-t(2\pi i\tilde{\lambda}_j + \frac{1}{2})} dt\, dx.$$

The integration can as well be extended over the set $\Omega = \{x \neq 0, x+e^{-t} \neq 0\}$, which we decompose into the three sets $\Omega_1 = \{x > 0\}$, $\Omega_2 = \{-e^{-t} < x < 0\}$, and $\Omega_3 = \{x < -e^{-t}\}$. Let $c = c_1 + c_2 + c_3$ the corresponding decomposition of the integral. Consider the map

$$\begin{aligned} \psi : \Omega &\to \mathbb{R}^2 \\ (x,t) &\mapsto \left(\underbrace{\log|x + e^{-t}|}_{=a}, \underbrace{\log|x|}_{=b} \right). \end{aligned}$$

Writing $\log|x| = \frac{1}{2}\log(x^2)$ we see that ψ is dfferentiable and we compute its functional determinant $|\det D\psi| = e^{-t-a-b}$.

Let first $(x,t) \in \Omega_1$. Then $e^b = x$ and $e^a = x + e^{-t}$, so that $e^{-t} = e^a - e^b$ and $|\det D\psi| = e^{-a-b}(e^a - e^b)$. Further, ψ is a diffeomorphism from Ω_1 to $\{a > b\}$ and so we see that c_1 equals

$$\int_{\{a>b\}} \phi(a,b) e^{a+b} (e^a - e^b)^{2\pi i\tilde{\lambda}_j - 1/2} da\, db.$$

Analogously we get

$$c_2 = \int_{\mathbb{R}}^2 \phi(a,b) e^{a+b} (e^a + e^b)^{2\pi i\tilde{\lambda}_j - 1/2} da\, db,$$

and

$$c_3 = \int_{\{a<b\}} \phi(a,b) e^{a+b} (e^b - e^a)^{2\pi i\tilde{\lambda}_j - 1/2} da\, db.$$

The proposition follows. □

Lemma 4.2 *The sequence $c(\eta, \eta_j) a_0^{\eta_j,\gamma} \beta(\tilde{\lambda}_j)$ is of moderate growth.*

Proof: Let $\varphi_j \in \pi_j$ be the unique K-invariant function which on K takes the value 1. Let $\phi_j = \eta_j(\varphi_j)$. Then

$$a_0^{\eta_j,\gamma} \beta(\tilde{\lambda}_j) = I_0^\gamma(\eta_j(\varphi_j)) = I_o^\gamma(\phi_j).$$

By [11] the sup-norm of ϕ_j satisfies

$$\|\phi_j\|_\infty = O(\tilde{\lambda}_j^{(d-1)/4}).$$

So that

$$a_0^{\eta_j,\gamma}\beta(\tilde{\lambda}_j) = O(\tilde{\lambda}_j^{(d-1)/4}).$$

Weyl's asymptotic law says that

$$\#\{j : |\tilde{\lambda}_j| \le x\} \ \sim\ cx^d$$

for some $c > 0$. This implies that the sequence $a_0^{\eta_j,\gamma}\beta(\tilde{\lambda}_j)$ is of moderate growth. For the numbers $c(\eta,\eta_j)$ it has been shown in [2], that

$$\sum_{|\lambda_j|\le T} |c(\eta,\eta_j)|^2 \ \le\ CT^d$$

for some $C > 0$. (The sequence $c(\eta,\eta_j)$ is named a_j in [2].) This implies that $c(\eta,\eta_j)$ is of moderate growth. $\square$

We now plug special functions into the formula of the proposition. Using convolution one constructs s function $\psi \in C_c^\infty(\mathbb{R})$ such that ψ and $\hat{\psi}$ are both non-negative functions and such that $\hat{\psi}(\xi) \ge 1$ for $|\xi| \le 1$. Here $\hat{\psi}$ denotes the Fourier transform,

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x)\, e^{-2\pi i x\xi} dx.$$

Give such ψ, let $T > 0$ and define $\phi = \phi_T \in C_c^\infty(\mathbb{R})$ by

$$\phi(s,t) = T\psi(T(s-t))\psi\left(\frac{s+t}{2}\right).$$

A similar test function was used in [10]. We compute

$$\int_{\mathbb{R}^2} \phi(s,t)e^{-2\pi i(s-t)(\tilde{\lambda}+k\tilde{\lambda}_\gamma)-\frac{s+t}{2}} dt\, ds = \hat{\psi}(1/2\pi i)\hat{\psi}\left(\frac{\tilde{\lambda}+k\tilde{\lambda}_\gamma}{T}\right).$$

So we get

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}\\ |\lambda+k\lambda_\gamma|\le T}} |a_k^{\eta,\gamma}|^2 \ &\le\ \sum_{k\in\mathbb{Z}} |a_k^{\eta,\gamma}|^2\, \hat{\psi}\left(\frac{\tilde{\lambda}+k\tilde{\lambda}_\gamma}{T}\right). \\ &=\ \hat{\psi}(1/2\pi i)^{-1} \sum_{k\in\mathbb{Z}} |a_k^{\eta,\gamma}|^2 \int_{\mathbb{R}^2} \phi(s,t)e^{-2\pi i(s-t)(\tilde{\lambda}+k\tilde{\lambda}_\gamma)-(s+t)\frac{1}{2}} dt\, ds,\end{aligned}$$

and this equals $\hat{\psi}(1/2\pi i)^{-1}$ times

$$2\sum_{j=1}^{\infty} c(\eta,\eta_j)a_0^{\eta_j,\gamma}\beta(\tilde{\lambda}_j)\times$$

$$\int_{\mathbb{R}^2}\phi(a,b)e^{a+b}\left(\left(e^a+e^b\right)^{2\pi i\tilde{\lambda}_j-\frac{1}{2}}+|e^a-e^b|^{2\pi i\tilde{\lambda}_j-\frac{1}{2}}\right)\,da\,db.$$

Analogous to the above we compute

$$\int_{\mathbb{R}^2}\phi(a,b)e^{a+b}\left(\left(e^a+e^b\right)^{2\pi i\tilde{\lambda}_j-\frac{1}{2}}+|e^a-e^b|^{2\pi i\tilde{\lambda}_j-\frac{1}{2}}\right)\,da\,db$$

$$=\hat{\psi}\left(-\tilde{\lambda}_j-\frac{1}{4\pi i}\right)\times$$

$$\int_{\mathbb{R}}\psi(a)\left((2\cosh(a/2T))^{2\pi i\tilde{\lambda}_j-1/2}+(2\sinh(|a|/2T))^{2\pi i\tilde{\lambda}_j-1/2}\right)\,da\,db.$$

$$=\hat{\psi}\left(-\tilde{\lambda}_j-\frac{1}{4\pi i}\right)\;\times\;O(T^{1/2})$$

as $T\to\infty$ uniformly in j. As $\hat{\psi}\left(-\tilde{\lambda}_j-\frac{1}{4\pi i}\right)$ is rapidly decreasing, the $d=2$ part of Theorem 2.4 follows. □

4.2 The summation formula in the case $d\geq 3$

Let us now consider the case $d\geq 3$. Setting $r^2=x_2^2+\cdots+x_{d-1}^2$ and using polar co-ordinates, we see that $R_\phi(\lambda_j)$ equals $\mathrm{vol}(B_{d-2})(d-2)$ times

$$\int_{\mathbb{R}^2}\int_0^\infty r^{d-3}\phi\left(\frac{1}{2}\log\left((x+e^{-t})^2+r^2\right),\frac{1}{2}\log\left(x^2+r^2\right)\right)\,e^{-t\left(2\pi i\tilde{\lambda}_j+\frac{d-1}{2}\right)}dr\,dt\,dx.$$

Here B_{d-2} is the solid ball of radius 1 in $\mathbb{R}^{d-2}$.

Consider the map

$$\begin{aligned}\psi:\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}&\to\mathbb{R}^3\\ \psi(x,r,t)&=\left(\underbrace{\frac{1}{2}\log\left((x+e^{-t})^2+r^2\right)}_{=a},\underbrace{\frac{1}{2}\log\left(x^2+r^2\right)}_{=b},t\right)\end{aligned}$$

A computation shows for the Jacobian:

$$\det(D\psi) = \frac{re^{-t}}{((x+e^{-t})^2+r^2)(x^2+r^2)} = re^{-(t+2a+2b)}.$$

The map ψ is a diffeomorphism onto the open set

$$\Omega = \{(a,b,t) \in \mathbb{R}^3 : (e^b - e^{-t})^2 < e^{2a} < (e^b + e^{-t})^2\}.$$

For $z \in \mathbb{C}$ let

$$K_z(a,b) = \int_{\mathbb{R}} \mathbb{1}_\Omega(a,b,t) \left[e^{2b} - \left(\frac{e^{2a} - e^{2b} - e^{-2t}}{2e^{-t}} \right)^2 \right]^{\frac{d-4}{2}} e^{2a+2b-t(z+\frac{d-3}{2})} dt.$$

We note that $(a,b,t) \in \Omega$ is equivalent to $-\log(e^a+e^b) < t < -\log(|e^a-e^b|)$, so that we can write

$$K_z(a,b) = \int_{-\log(e^a+e^b)}^{-\log(|e^a-e^b|)} \left[e^{2b} - \left(\frac{e^{2a} - e^{2b} - e^{-2t}}{2e^{-t}} \right)^2 \right]^{\frac{d-4}{2}} e^{2a+2b-t(z+\frac{d-3}{2})} dt.$$

For $a \neq b$ this integral converges absolutely for all z. We conclude that

$$R_\phi(\lambda_j) = (d-2)\mathrm{vol}(B_{d-2}) \int_{\mathbb{R}^2} \phi(a,b) K_{2\pi i \tilde{\lambda}_j}(a,b)\, da\, db.$$

Summarising, we get the following summation formula.

Theorem 4.3 *(Summation Formula) Let ϕ be a smooth function of compact support on $\mathbb{R}^2$. For $d \geq 3$ the expression*

$$\sum_{k\in\mathbb{Z}} |a_k^{\eta,\gamma}|^2 \int_{\mathbb{R}^2} \phi(s,t) e^{-2\pi i(s-t)(\tilde{\lambda}+k\tilde{\lambda}_\gamma)-(s+t)\frac{d-1}{2}} dt\, ds$$

equals

$$(d-2)\mathrm{vol}(B_{d-2}) \sum_{j=1}^{\infty} c(\eta,\eta_j) a_0^{\eta_j,\gamma} \beta(\tilde{\lambda}_j) \int_{\mathbb{R}^2} \phi(a,b) K_{2\pi i\tilde{\lambda}_j}(a,b)\, da\, db.$$

Again we set
$$\phi(a,b) = T\psi(T(a-b))\psi((a+b)/2),$$
and we get that $\int_{\mathbb{R}^2} \phi(a,b) K_{2\pi i y}(a,b)\, da\, db$ equals $\hat{f}_T(y)$, where

$$f_T(t) = \int_{\{2e^b \sinh(|a|/2) < e^{-t} < 2e^b \cosh(a/2)\}} TH(Ta,b,t)\, da\, db,$$

and

$$H(a,b,t) = \psi(a)\psi(b) \left[e^{2b-a} - \left(\frac{e^{2b+a} - e^{2b-a} - e^{-2t}}{2e^{-t}} \right)^2 \right]^{\frac{d-4}{2}} e^{4b-t(d-3)/2}.$$

We claim that f_T is a Schwartz function, i.e., that for every $N, M \geq 0$ there exists $C_{M,N,T} > 0$ such that

$$|f_T^{(N)}(t)| \leq C_{M,N,T}(1+|t|)^M.$$

Moreover, for $N \geq 1$, the constant $C_{M,N,T}$ can be chosen independent of T.

The claim is immediate for $N = 0$, but it is not obvious, why f_T is differentiable. For this write $H_T(a,b,t) = TH(Ta,b,t)$, and for $h \neq 0$ consider the difference quotient $\frac{f(t+h)-f(t)}{h}$, which equals

$$\frac{1}{h}\int_{\dots < e^{-t-h} < \dots} H_T(a,b,t+h) da\, db - \frac{1}{h}\int_{\dots < e^{-t} < \dots} H_T(a,b,t)\, da\, db$$
$$= \frac{1}{h}\int_{\dots < e^{-t-h} <} H_T(a,b,t+h) - H_T(a,b,t) da\, db$$
$$+ \frac{1}{h}\int_{\dots < e^{-t-h} < \dots} - \int_{\dots < e^{-t} < \dots} H_T(a,b,t) da\, db.$$

As $h \to 0$, the first line tends to

$$\int_{\dots < e^{-t} < \dots} H_T'(a,b,t)\, da\, db,$$

where H_T' is the t-derivative. For $h > 0$, the second line equals

$$\frac{1}{h}\int_{2e^b \sinh(|a|/2) < e^{-t-h} < 2e^b \cosh(a/2) < e^{-t}} H_T(a,b,t)\, da\, db$$

$$-\frac{1}{h}\int_{e^{-t-h}<2e^b\sinh(|a|/2)<e^{-t}<2e^b\cosh(a/2)} H_T(a,b,t)\,da\,db.$$

The first of these two summands tends to

$$e^{-t}\int_{\mathbb{R}} H_T\left(a, \log\left(\frac{1}{2}\frac{e^{-t}}{\cosh(a/2)}\right), t\right)\,da$$

which equals

$$e^{-t}\int_{\mathbb{R}} H\left(a, \log\left(\frac{1}{2}\frac{e^{-t}}{\cosh(a/2T)}\right), t\right)da,$$

and the second to

$$-e^{-t}\int_{\mathbb{R}} H_T\left(a, \log\left(\frac{1}{2}\frac{e^{-t}}{\sinh(|a|/2)}\right), t\right)\,da,$$

and this is

$$-e^{-t}\int_{\mathbb{R}} H\left(a, \log\left(\frac{1}{2}\frac{e^{-t}}{\sinh(|a|/2T)}\right), t\right)\,da.$$

The case $h<0$ is dealt with in an analogous fashion and yields the same limits. From this the claim follows easily. The claim implies that $\hat{f}_T$ is rapidly decreasing, independent of T, which implies the second part of Theorem 2.4.

□

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